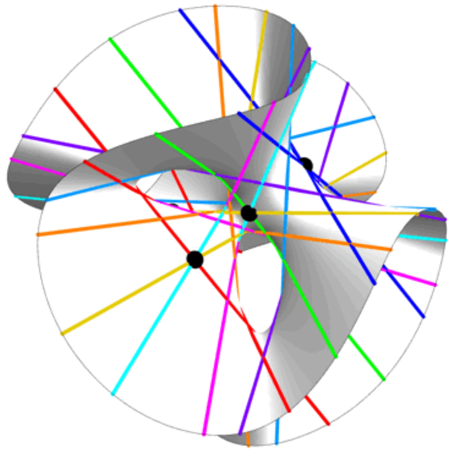
11. The 27 lines on a smooth cubic surface

This chapter

Use all tools developed so far to study set of lines inside a smooth cubic surface $X \subseteq \mathbb{P}^3$.



27 Lines on a Cubic Surface – Greg Egan

Goals

enumerative geometry

- show that the number of such lines is exactly 27
- study their configuration
- prove that X is birational to $\mathbb{P}^2$

↘ birational geometry of hypersurfaces

Note: · results not used in later chapters ⇝ proofs not in all details
· restrict to $K = \mathbb{C}$ for simplicity

Lem The Fermat cubic $X = V(x_0^3 + x_1^3 + x_2^3 + x_3^3) \subseteq \mathbb{P}^3$ contains exactly 27 lines.

Pf $G(2,4)$ = space of all lines $L \subseteq \mathbb{P}^3$
⇝ after permuting coordinates: Plücker coord. $z_{12} \neq 0$

$$\rightsquigarrow \quad L = \text{row-span}\begin{pmatrix} 1 & 0 & a_2 & a_3 \\ 0 & 1 & b_2 & b_3 \end{pmatrix}$$

[Note: before looked at column spans]

$$= \{(s : t : a_2 s + b_2 t : a_3 s + b_3 t) : (s:t) \in \mathbb{P}^1\}$$

$$L \subseteq X \iff s^3 + t^3 + (a_2 s + b_2 t)^3 + (a_3 s + b_3 t)^3 \equiv 0 \quad \forall s,t$$

↓ expand in s,t

$$a_2^3 + a_3^3 = -1, \quad a_2^2 b_2 = -a_3^2 b_3, \quad a_2 b_2^2 = -a_3 b_3^2, \quad b_2^3 + b_3^3 = -1.$$

$$\underbrace{a_2^3+a_3^3=-1}_{(1)},\quad \underbrace{a_2^2 b_2=-a_3^2 b_3}_{(2)},\quad \underbrace{a_2 b_2^2=-a_3 b_3^2}_{(3)},\quad \underbrace{b_2^3+b_3^3=-1}_{(4)}.$$

If $a_2,a_3,b_2,b_3 \neq 0 \rightsquigarrow (2)^2/(3) \Rightarrow a_2^3=-a_3^3$ $\lightning$ to (1)

Reordering: $\boxed{a_2=0} \rightsquigarrow (1): a_3^3=1,\ (2): b_3=0,\ (4): b_2^3=-1$

$\rightsquigarrow (3)$: automatic

$\Rightarrow$ get 9 lines: $a_3=-\omega^j,\ b_2=-\omega^k,\ a_2=b_3=0,$

for $\omega=\exp(2\pi i/3),\ j,k\in\{0,1,2\}$

All possible permutations of coordinates! other combinat. possib: equivalent via row operat.

$$\underbrace{\begin{pmatrix}1&0&0&-\omega^j\\0&1&-\omega^k&0\end{pmatrix}}_{9},\ \underbrace{\begin{pmatrix}1&0&-\omega^j&0\\0&1&0&-\omega^k\end{pmatrix}}_{9},\ \underbrace{\begin{pmatrix}1&-\omega^j&0&0\\0&0&1&-\omega^k\end{pmatrix}}_{9}$$

$\rightsquigarrow 27=9+9+9$ $\square$

Cor $X\subseteq \mathbb{P}^3$ Fermat cubic

(a) $\forall L\subseteq X$ line $\exists$ exactly 10 other lines $L'\subseteq X: L\cap L'\neq\emptyset$

(b) $\forall L_1,L_2\subseteq X$ disjoint lines $\exists$ exactly 5 other lines L': $L'\cap L_1\neq\emptyset\neq L'\cap L_2$

Pf (a) After permuting & scaling coord.

$\rightsquigarrow L=$ row span $\begin{pmatrix}1&0&0&-1\\0&1&-1&0\end{pmatrix}$

$\rightsquigarrow$ lines L' meeting L:

4 lines as the row span of $\begin{pmatrix}1&0&0&-\omega^j\\0&1&-\omega^k&0\end{pmatrix}$ for $(j,k)=(1,0),(2,0),(0,1),(0,2)$,

3 lines as the row span of $\begin{pmatrix}1&0&-\omega^j&0\\0&1&0&-\omega^k\end{pmatrix}$ for $(j,k)=(0,0),(1,2),(2,1)$,

3 lines as the row span of $\begin{pmatrix}1&-\omega^j&0&0\\0&0&1&-\omega^k\end{pmatrix}$ for $(j,k)=(0,0),(1,2),(2,1)$.

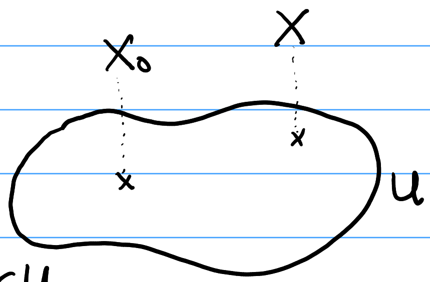
(b) similar.

$\square$

The moduli space of smooth cubic surfaces

Strategy

- counted the number of lines on Fermat cubic $X_0 = V(x_0^3 + \dots + x_3^3)$ by hand $\rightsquigarrow 27$
- construct algebraic variety $\mathcal{U}$ parameterizing all smooth cubic surfaces X $\rightsquigarrow$ moduli space
- show that (#lines on X) is a locally constant function $X \to \mathbb{Z}$
- $\mathcal{U}$ connected $\Rightarrow$ #lines on $X = 27$ $\forall X \in \mathcal{U}$.



Cubic surface $X = V_{\mathbb{P}}(f) \subseteq \mathbb{P}^3$ described by its equation:

$$f = a_0 \cdot x_0^3 + a_1 \cdot x_0^2 x_1 + \dots + a_{19} \cdot x_3^3 \qquad \rightsquigarrow \binom{3+3}{3} = 20$$

Want: $f \neq 0$ and only care about f up to scaling $\left(V_{\mathbb{P}}(\lambda f) = V_{\mathbb{P}}(f)\right)$ $\lambda \in k^\times$

Def The moduli space of cubic equations is given by

$$\mathcal{E}_{3,3} = \mathbb{P}^{19}$$

Exercise Moduli $\mathcal{E}_{d,n}$ of degree d equations in $\mathbb{P}^n$?

Exa Fermat cubic

$$X_0 = V(\underbrace{x_0^3 + \dots + x_3^3}_{f_C}) \mathrel{\hat=} C = (\underset{x_0^3}{1} : 0 : \dots : 0 : \underset{x_1^3}{1} : 0 : \dots : \underset{x_2^3}{1} : \dots : \underset{x_3^3}{1}) \in \mathbb{P}^{19}$$

Universal surface

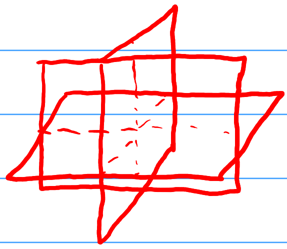
$$\mathcal{X} = \{([f], x) \in \mathcal{E}_{3,3} \times \mathbb{P}^3 : f(x) = 0\} \subseteq \mathcal{E}_{3,3} \times \mathbb{P}^3$$

$$\begin{array}{c} \mathcal{X} \\ \pi \big\downarrow \\ \mathcal{E}_{3,3} \end{array} \qquad \begin{array}{c} ([f], x) \\ \downarrow \\ [f] \end{array}$$

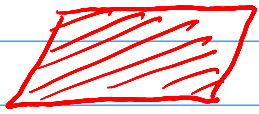
universal family

Then $\pi^{-1}([f]) = V_{\mathbb{P}}(f) \subseteq \{[f]\} \times \mathbb{P}^3$

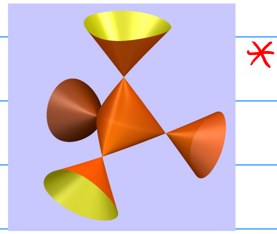
Problem $\mathcal{E}_{3,3}$ contains many $[f]$ not giving smooth cubics:



$f = x_0 \cdot x_1 \cdot x_2$



$f = x_0^3$



$f = x_0x_1x_2 + x_1x_2x_3 + x_2x_3x_0 + x_3x_0x_1$

Cayley surface

$\ast$ By Salix alba - Own work, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=12349290

Solution

$$\begin{array}{l} \mathcal{X} \overset{\text{closed}}{\supseteq} \mathcal{X}^{\text{sing}} = \left\{ ([f], x) : \underbrace{J_f}_{\text{Jacobian}}(x) = \left(\frac{\partial f}{\partial x_i}\Big|_x \right) = 0 \right\} \subseteq \mathcal{E}_{3,3} \times \mathbb{P}^3 \\ \pi \big\downarrow \\ \mathcal{E}_{3,3} \xleftarrow{\quad \tilde{\pi} \quad} \end{array}$$

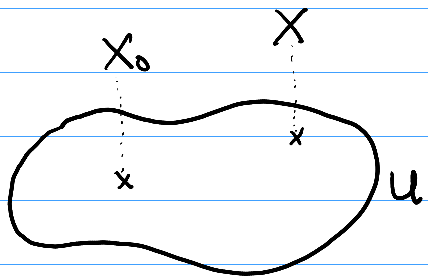
$\mathbb{P}^3$ complete $\Rightarrow$ $\tilde{\pi}$ closed map

$\Rightarrow \tilde{\pi}(\mathcal{X}^{\text{sing}}) = \{ [f] \in \mathcal{E}_{3,3} : f \text{ has singularity} \}$

closed in $\mathcal{E}_{3,3}$

Def $\mathcal{U} = \mathcal{E}_{3,3} \setminus \tilde{\pi}(\mathcal{X}^{\text{sing}})$ <u>moduli space of smooth cubic surfaces</u>

<u>Note</u>: $\underset{\mathbb{P}^{19}}{\underset{\wr\!\wr}{\mathcal{E}_{3,3}}}$ irreducible $\Rightarrow$ $\mathcal{U}$ irreducible, hence connected



<u>Intuition</u> can deform any cubic X to Fermat cubic X_0

The incidence correspondence of lines in cubic surfaces

Have constructed

$\mathcal{U} \subseteq \mathbb{P}^{19}$ moduli space of smooth cubic surfaces

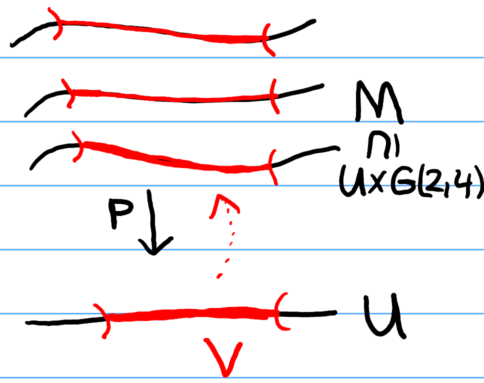
Coordinates: $c = (c_\alpha)_{\substack{\alpha \in \mathbb{N}^4 \\ \sum \alpha_i = 3}} \in \mathcal{U} \rightsquigarrow f_c = \sum_\alpha c_\alpha \cdot x^\alpha \rightsquigarrow X = V_p(f_c)$

Next: lines on cubic surfaces

Construction (Incidence correspondence)

$$\begin{array}{l} M := \{(X,L) : L \text{ is a line contained in } X\} \subseteq \mathcal{U} \times G(2,4) \\ p \downarrow \quad \begin{smallmatrix}(X,L)\\ \downarrow \\ X\end{smallmatrix} \\ \mathcal{U} \end{array}$$

Lem The incidence correspondence M is
(a) closed in the Zariski topology of $\mathcal{U} \times G(2,4)$
(b) locally in the classical topology on M the graph of a continuous differentiable function $\mathcal{U} \supseteq V \longrightarrow G(2,4)$



Pf (a) Check on $U_0 = \{L : L = \text{row span}\begin{pmatrix} 1 & 0 & a_2 & a_3 \\ 0 & 1 & b_2 & b_3 \end{pmatrix}\} \subseteq G(2,4)$

$(c,a,b) = ((c_\alpha)_\alpha, a_2, a_3, b_2, b_3) \in \mathcal{U} \times U_0$

$$\begin{aligned}
(c,a,b) \in M &\iff f_c(s\cdot(1,0,a_2,a_3) + t\cdot(0,1,b_2,b_3)) = 0 \ \forall s,t \\
&\iff \sum_\alpha c_\alpha \cdot s^{\alpha_0} \cdot t^{\alpha_1} \cdot (s a_2 + t b_2)^{\alpha_2} \cdot (s a_3 + t b_3)^{\alpha_3} = 0 \ \forall s,t \\
&\iff \sum_i s^i t^{3-i} \cdot F_i(c,a,b) = 0 \ \forall s,t \\
&\iff F_i(c,a,b) = 0 \ \forall i = 0,\dots,3 \rightsquigarrow \text{closed in } \mathcal{U} \times U_0
\end{aligned}$$

(b) $G = PGL(4) \curvearrowright \mathbb{P}^3$ Projective automorphisms

$$\begin{array}{lll} \rightsquigarrow & G \curvearrowright M & g.(X,L) = (gX, gL) \quad , X, L \subseteq \mathbb{P}^3 \\ & \quad\ \downarrow p & \\ & G \curvearrowright \mathcal{U} & g.X = gX \end{array}$$

For $(X,L) \in M$ $\exists g \in G$: $g.L = \mathrm{Lin}(e_1, e_2) \in U_0 \subseteq G(2,4)$

$\rightsquigarrow$ suffices to prove (b) for $L = \mathrm{Lin}(e_1,e_2) \rightsquigarrow \left.\begin{pmatrix} 1 & 0 & a_2 & a_3 \\ 0 & 1 & b_2 & b_3 \end{pmatrix}\right|_{a_2=a_3=b_2=b_3=0}$

$G(2,4) \supseteq \underbrace{U_0 \cong \mathbb{A}^4}_{\substack{(a_2,a_3,b_2,b_3) \\ 4 \text{ variables}}}$ and $M \subseteq \mathcal{U} \times U_0$ cut out by $\underbrace{F_i(c,a,b) = 0}_{\substack{i=0,1,2,3 \\ 4 \text{ equations}}}$

Implicit function theorem: suffices to check that

$$J = \frac{\partial(F_0, F_1, F_2, F_3)}{\partial(a_2, a_3, b_2, b_3)} \quad \text{invertible at } a = b = 0.$$

<u>Computing J</u>

$$\frac{\partial}{\partial a_2}\left(\sum_i s^i t^{3-i} F_i\right)\Big|_{a=b=0} = \frac{\partial}{\partial a_2} f_c(s,t, sa_2 + tb_2, sa_3 + bt_3)\Big|_{a=b=0}$$

$$= s \cdot \frac{\partial f_c}{\partial x_2}(s,t,0,0)$$

$\hookrightarrow$ (s,t)-coefficients = first column of J

Other columns: $s \cdot \frac{\partial f_c}{\partial x_3}(s,t,0,0)$, $t \cdot \frac{\partial f_c}{\partial x_2}(s,t,0,0)$, $t \cdot \frac{\partial f_c}{\partial x_3}(s,t,0,0)$.

J not invertible $\Rightarrow$ $\exists$ linear relation between columns:

$$\underbrace{(\lambda_2 s + \mu_2 t) \cdot \frac{\partial f_c}{\partial x_2}(s,t,0,0)}_{} + \underbrace{(\lambda_3 s + \mu_3 t) \frac{\partial f_c}{\partial x_3}(s,t,0,0)}_{} = 0 \ \forall s,t$$

homogeneous degree 3 in s,t $\Rightarrow$ $\frac{\partial f_c}{\partial x_2}, \frac{\partial f_c}{\partial x_3}$ have common zero $q = (q_0, q_1, 0, 0) \in L$

Summary: $\exists q \in L = \mathrm{Lin}(e_1, e_2)$: $\frac{\partial f_c}{\partial x_2}(q) = \frac{\partial f_c}{\partial x_3}(q) = 0$

$L \subseteq X = V_P(f_c) \Rightarrow f_c(s,t,0,0) = 0 \ \forall s,t$

$$\xRightarrow{\partial_s, \partial_t} \frac{\partial f_c}{\partial x_0}(q) = \frac{\partial f_c}{\partial x_1}(q) = 0$$

Jacobi criterion $\Rightarrow q \in X$ is singular point $\lightning$ $\square$

Lines on (cubic) surfaces

Have seen

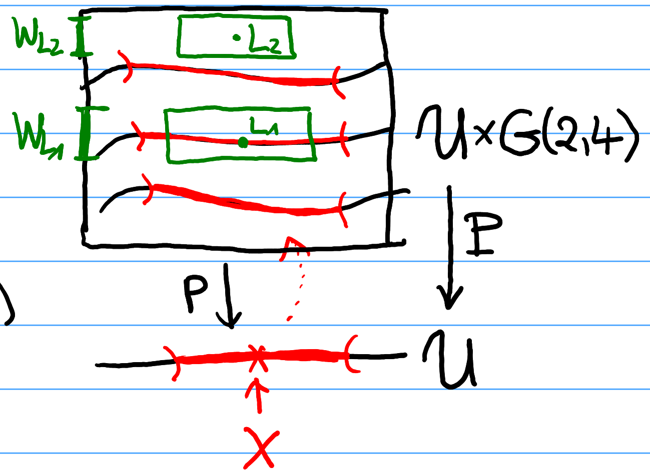
$$M = \{(X,L) : L \text{ lies in } X\} \subseteq \mathcal{U} \times G(2,4)$$
$$\downarrow p \qquad \nwarrow \text{locally} = \text{graph of different. function } \mathcal{U} \supseteq V \to G(2,4).$$
$$\mathcal{U}$$

Cor Every smooth cubic surface contains exactly 27 lines.

Pf Use complex / classical topology everywhere

$X \in \mathcal{U}$ fixed cubic surface

$L \in G(2,4)$ line in $\mathbb{P}^3$



Case 1 $L = L_1$ lies in X

$\Longrightarrow \exists$ open nbhd. $V_L \times W_L$ of (X,L):

$M \cap V_L \times W_L = $ graph of $V_L \to G(2,4)$

$\rightsquigarrow$ every cubic in V_L contains exactly one line in W_L

Case 2 $L = L_2$ does not lie in X

$\Longrightarrow \exists$ open nbhd. $V_L \times W_L$ of (X,L):

$M \cap V_L \times W_L = \emptyset \quad \leftarrow M$ closed in $\mathcal{U} \times G(2,4)$

Fiber $P^{-1}(X) \cong G(2,4)$ covered by $\{W_L : L \in G(2,4)\}$

$G(2,4)$ compact $\Rightarrow$ exists a finite subcover L_j

On $V = \bigcap_j V_{L_j} \subseteq \mathcal{U}$:

any $X \in V$ has same # of lines $= \# L_j$ in case 1 above

$\Rightarrow \mathcal{U} \xrightarrow{h} \mathbb{N}, \; X \mapsto \#\{\text{lines in } X\}$

is locally constant function.

To conclude:

$\mathcal{U} = \mathbb{P}^{19} \setminus Z$

$\uparrow$ proper Zariski closed subset

$\Rightarrow$ $\mathbb{C}$-codimension ≥ 1

$\Rightarrow$ $\mathbb{R}$-codimension ≥ 2 $\Rightarrow$ $\mathcal{U}$ connected $\square$

Rmk

(a) $p: M \to \mathcal{U}$ is 27-sheeted covering map

(b) X cubic surface

$\Rightarrow \exists!$ 10 lines meeting a given one,

5 lines meeting any two disjoint ones

Proof via suitable incidence correspondences of (X, L_1, L_2)

(X, L_1, L_2, L_3).

Rmk (Lines in surfaces of other degrees)

Q Why count lines in degree $d=3$ surfaces?

Dimension count:

- $G(2,4)$: dimension 4, local coord. a_2, a_3, b_2, b_3
- Given $f \in K[x_0, \dots, x_3]$ of degree d

$\rightsquigarrow f|_L = f_c(s, t, sa_2 + tb_2, sa_3 + tb_3)$ degree d in s, t

- $f|_L = 0 \rightsquigarrow d+1$ equations in a_2, a_3, b_2, b_3

$\Rightarrow$ expect : $d < 3 \rightsquigarrow$ infinitely many lines

$d = 3 \rightsquigarrow$ finite number (27 ?)

$d > 3 \rightsquigarrow$ general degree d surface X has no lines

Birational geometry of cubic surfaces

Have seen $C \subseteq \mathbb{P}^2$ smooth quadric $\Rightarrow C \xrightarrow{\sim} \mathbb{P}^1$ projection from point

Does something similar work for $X \subseteq \mathbb{P}^3$ smooth cubic ?

Guess: $X \cong \mathbb{P}^2$? No: $\cdot$ X contains two disjoint lines
$\cdot$ any two curves in $\mathbb{P}^2$ meet [EX6.31(b)]

But almost!

Pro Any smooth cubic surface X is birational to $\mathbb{P}^2$.

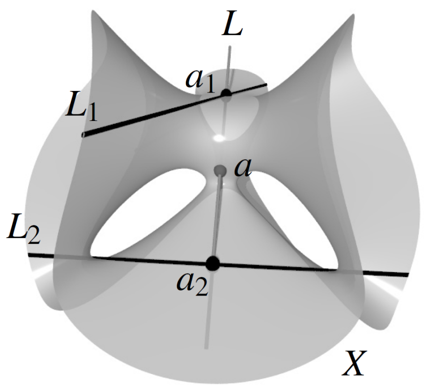
Pf X contains two disjoint lines L_1, L_2

$\rightsquigarrow$ give mutually inverse rational maps

$X \dashrightarrow L_1 \times L_2$ and $L_1 \times L_2 \dashrightarrow X$

$\Rightarrow X \simeq L_1 \times L_2 \cong \mathbb{P}^1 \times \mathbb{P}^1 \simeq \mathbb{P}^2$

birational



Exercise: $L_1, L_2 \subseteq \mathbb{P}^3$ disjoint lines,
$a \in \mathbb{P}^3 \setminus (L_1 \cup L_2)$

$\Rightarrow \exists!$ line L through a meeting L_1, L_2

$X \overset{f}{\dashrightarrow} L_1 \times L_2$: send $a \in X \setminus (L_1 \cup L_2)$ to pair (a_1, a_2) of intersection points $(L \cap L_1, L \cap L_2)$

$L_1 \times L_2 \overset{g}{\dashrightarrow} X$: send (a_1, a_2) to third intersection point of line $L = L_{a_1 a_2}$ with X
(well-defined if $L \nsubseteq X$). $\square$

Pro Any smooth cubic surface X is isomorphic to $\mathbb{P}^2$ blown up at 6 points.

Pf sketch (more details in [Gathmann])

- show f extends to morphism $f: X \longrightarrow L_1 \times L_2 = \mathbb{P}^1 \times \mathbb{P}^1$
- inverse map g defined away from pts (a_1, a_2) w/ $L_{a_1 a_2} \subseteq X$
- Check: $X =$ blow-up of 5 pts on $\mathbb{P}^1 \times \mathbb{P}^1$ Know: $\exists!$ 5 lines meeting L_1, L_2

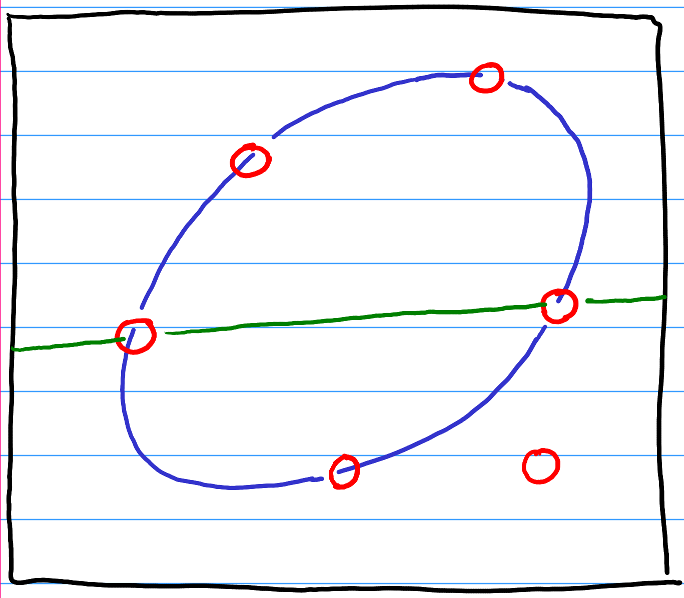
$\overset{\text{[Lem 9.27]}}{=}$ blow-up of 6 pts on $\mathbb{P}^2$.

$Bl_{p_1} \mathbb{P}^1 \times \mathbb{P}^1 \cong Bl_{q_1, q_2} \mathbb{P}^2$

$\square$

Interpretation of the 27 lines

$X = Bl_{P_1,\dots,P_6}\mathbb{P}^2$ cubic surface



6 exceptional hypersurfaces E_a

$15 = \binom{6}{2}$ strict transforms L_{ij} of lines through P_i, P_j

$6 = \binom{6}{5}$ strict transforms Q_I of conic through $P_{i_1}, \dots, P_{i_5}$

$6 + 15 + 6 = 27$

Every line meets 10 others:

- E_a meets 5 lines L_{ja} and 5 lines Q_I $(a \in I)$
- L_{ij} meets
  - $\to$ E_i, E_j except. hypersurfaces over i, j
  - $\to$ $\binom{4}{2} = 6$ remaining lines L_{km}
  - $\to$ 2 conics $Q_{\{1,\dots,6\}\setminus\{i\}}$ and $Q_{\{1,\dots,6\}\setminus\{j\}}$
- Q_I with $I = \{1,\dots,6\}\setminus\{b\}$
  - $\to$ 5 exc. hypersurfaces E_a $(a \in I)$
  - $\to$ 5 lines L_{ab} $(a \in I)$